Updated Dec 6, 2023

1. Evaluate each indefinite integral

(a) $\int -9x^2 + 10\ dx$ $= -3x^3 + 10x + C$

(b) $\int \frac{15}{x^4} + \frac{8}{x^5}\ dx$ $= \int 15x^{-4} + 8x^{-5}\ dx = \frac{-5}{x^3} - \frac{2}{x^4} + C$

(c) $\int \frac{-5\sqrt[3]{x^2}}{3}\ dx$ $= \frac{-5}{3}\int x^{2/3}\ dx = -\frac{5}{3}\left(\frac{3}{5}x^{5/3}\right) + C = -x^{5/3} + C$

(d) $\int \frac{x^4+1}{3x^2}\ dx$ $= \frac{1}{3}\int x^2 + x^{-2}\ dx = \frac{1}{3}\left(\frac{x^3}{3} - \frac{1}{x}\right) + C = \frac{x^3}{9} - \frac{1}{3x} + C$

(e) $\int \frac{x^2+x+1}{\sqrt{x}}\ dx$ $= \int x^{3/2} + x^{1/2} + x^{-1/2}\ dx = \frac{2}{5}x^{5/2} + \frac{2}{3}x^{3/2} + 2x^{1/2} + C$

(f) $\int 2x\sqrt{1-x^2}\ dx$ $= -\int u^{1/2}\ du = -\frac{2}{3}(1-x^2)^{3/2} + C$

$\boxed{\begin{array}{l} u = 1-x^2 \\ -du = +2x\ dx \end{array}}$

(g) $\int x\sqrt{1-x}\ dx$

$\boxed{\begin{array}{l} u = 1-x \quad \text{so } x = 1-u \\ du = -dx \\ -du = dx \end{array}}$

$-\int (1-u)u^{1/2}\ du = -\int u^{1/2} - u^{3/2}\ du$

$= -\frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2} + C$

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C$

(h) $\int \sec^2 x\ dx$ $= \tan x + C$

(i) $\int \sin^3 x \cos x\ dx$ $= \int u^3\ du = \frac{u^4}{4} + C = \frac{1}{4}\sin^4 x + C$

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\ dx \end{array}}$

2. Evaluate each definite integral

(a) $\int_0^{\pi/2} \cos x \, dx$ $= \sin x \Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$

(b) $\int_0^{\pi/4} \sec^2 x \, dx$ $= \tan x \Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1$

(c) $\int_1^{3} 3x + 4 \, dx$ $= \frac{3x^2}{2} + 4x \Big|_1^{3} = \left(\frac{3(3)^2}{2} + 4(3)\right) - \left(\frac{3(1)^2}{2} + 4(1)\right)$

$= 25.5 - 5.5 = 20.$

3. Let $f(x) = x^2$. Use the limit definition of the definite integral to express $\int_5^9 f(x)\, dx$ (no need to evaluate it)

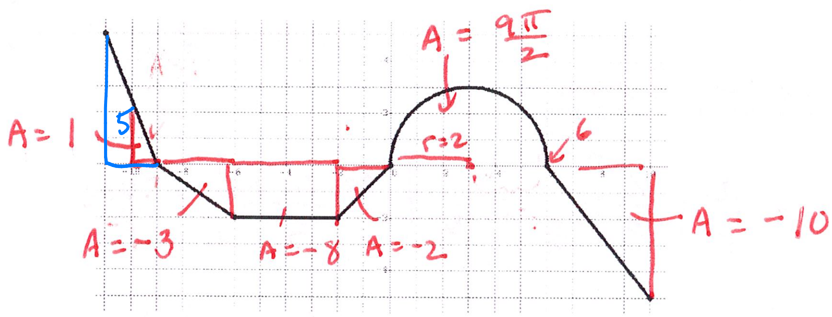
(a) If the base of the rectangle $\Delta x = \frac{b-a}{n}$, then here Δx is: $\frac{9-5}{n} = \frac{4}{n}$

(b) If $c_i = a + i(\Delta x)$, then here c_i is: $5 + \frac{4i}{n}$

(c) So the height of the rectangle $f(c_i)$ is: $\left(5 + \frac{4i}{n}\right)^2$

(d) Using sigma notation, the integral expressed as a limit is: $\lim_{n\to\infty} \sum_{i=\cancel{0}1}^{\cancel{\infty}\,n} \left(5+\frac{4i}{n}\right)^2\left(\frac{4}{n}\right)$

4. The graph of $f(x)$ below is made up of line segments and semi-circles as shown.



(a) $\int_{-11}^{10} f(x)\, dx =$ $5\cancel{1} - 3 - 8 - 2 + \frac{9\pi}{2} - 10$

(b) $\int_{10}^{0} f(x)\, dx =$ $-\int_0^{10} f(x) = -\left(\frac{9\pi}{2} - 10\right) = 10 - \frac{9\pi}{2}$

(c) $\int_{-2}^{-9} f(x)\, dx =$ $-\int_{-9}^{-2} f(x) = -(-3-8) = 11$

(d) $\int_3^3 f(x)\, dx =$ 0

(e) $\int_3^{\cancel{5}\,10} f(x)\, dx =$ $\frac{9\pi}{4} - 10$

5. Suppose the f and h are continuous functions and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$

(a) $\int_1^9 -2f(x)\,dx =$ $-2\int_1^9 f(x) = -2(-1) = 2$

(b) $\int_9^7 h(x) - f(x)\,dx =$ $-\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx = \cancel{-5} + \cancel{4} = \cancel{-1}$

$-4 + 5 = 1$

6. If $g(1) = 0$ and $\frac{dg}{dx} = 3x^2\sqrt{x^3+8}$ then $g(x) =$ $\int \frac{dg}{dx}\,dx = (x^3+8)^{3/2} - 27$

$u = x^3+8$

$du = 3x^2\,dx$

$= \int u^{1/2}\,du$

$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^3+8)^{3/2} + C$

$\frac{2}{3}(1+8)^{3/2} + C = 0, \quad C = -27 \cdot \frac{2}{3} = -18$

7. (4 points) Let $F(x) = \int_0^{x^2} \cos t^2\,dt$. Use the second fundamental theorem of calculus to find $F'(x)$.

$\cos(x^2)^2 \cdot 2x$ or $2x \cos x^4$

(research $+C$?)

8. What is the average value of $f(x) = x^2$ over the following intervals: even function

(a) $[-20, 20]$

$\frac{1}{20-(-20)}\int_{-20}^{20} x^2\,dx = \frac{2}{40}\left[\frac{x^3}{3}\right]_0^{20} = \frac{1}{20}\left[\frac{20\cdot20\cdot20}{3}\right] = \frac{400}{3}$

(b) $[0, 3]$

$\frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}\left[\frac{3^3}{3} - \frac{0^3}{3}\right] = 3$

(c) $[-1, 2]$

$\frac{1}{2-(-1)}\int_{-1}^{2} x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2} = \frac{1}{3}\left[\frac{2^3}{3} - \frac{(-1)^3}{3}\right] = 1$

9. If the average value of $g(x)$ on the interval $3 \le x \le k$ is 120, then $\int_3^k g(x)\,dx =$ $(k-3)120$

10. The function f is continuous on the closed interval $[0, 6]$ where x is measured in hours and $f(x)$ is measured in pounds. Selected values of $f(x)$ are shown in the table below.

x (hours)	0	2	4	6
$f(x)$ (pounds)	4	k	8	12

The trapezoidal approximation for $\int_0^6 f(x)\,dx$ found with three subintervals of equal length is 52.

(a) Using this approximation, estimate the average value of f between $x = 0$ and $x = 6$.
Hint: remember the units

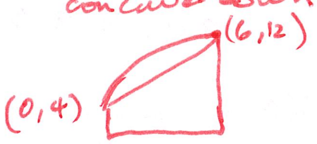
$\frac{52}{6-0} = \frac{26}{3}$ pounds

(b) What is the value of k?

$2\left[\frac{4+k}{2} + \frac{k+8}{2} + \frac{8+12}{2}\right] = 52 = 2k + 32; \; k = \frac{52-32}{2} = 10$ pounds

(c) If $f''(x) < 0$ for all x in $[0, 6]$, would this be an overestimate or an underestimate?

concave down



underestimate because f is concave down

11. (Calculator Active) A hot cup of coffee is taken into a classroom and set on a desk to cool. When $t = 0$, the temperature of the coffee is 113°F. The rate at which the temperature of the coffee is dropping is modeled by a differentiable monotonic function $R(t)$ for $0 \le t \le 8$, where $R(t)$ is measured in degrees Fahrenheit per minute and t is measured in minutes. Values of $R(t)$ at selected times are shown below

3 2 3

t (minutes)	0	3	5	8
$R(t)$ in (°F/minute)	5.5	2.7	1.6	0.8

ΔR -2.8 -1.1 $-.8$

(a) Estimate the temperature of the coffee at $t = 8$ minutes by using a left Riemann sum with three subintervals and values from the table. Show the computations that lead to your answer. Justify why you think the actual temperature of the coffee is hotter or cooler than this estimate.

$113 - [3(5.5) + 2(2.7) + 3(1.6)] = \cancel{83.6}$ 86.3 °F

left R.S on decreasing is overestimate of cooling so coffee is hotter than ~~83.6~~ 86.3

(b) Estimate the temperature of the coffee at $t = 8$ minutes by using a right Riemann sum with three subintervals and values from the table. Show the computations that lead to your answer. Justify why you think the actual temperature of the coffee is hotter or cooler than this estimate.

$113 - [3(2.7) + 2(1.6) + 3(0.8)] = 99.3$°F

rt R.S. on decr. is an underestimate of cooling so coffee is colder than 99.3

(c) Estimate the temperature of the coffee at $t = 8$ minutes by using a trapezoidal Riemann sum with three subintervals and values from the table. Show the computations that lead to your answer. Justify why you think the actual temperature of the coffee is hotter or cooler than this estimate.

$113 - \left[3\left(\frac{5.5+2.7}{2}\right) + 2\left(\frac{2.7+1.6}{2}\right) + 3\left(\frac{1.6+0.8}{2}\right)\right] = 92.8$°F

trap RS on concave up is an overestimate of cooling so coffee is hotter than 92.8°F

TI Tip: → set to Y_1

12. (Calculator Active: Always show the "math" you used, and the calculator's answer accurate to 3 decimal places. Avoid approximation before the last step.) Let $f(x) = 5\ln x$. Approximate $\int_1^5 f(x)\,dx$ using the following methods:

(a) Find the left Riemann sum using four subintervals of equal length. $\Delta x = \frac{5-1}{4} = 1$

$1\left(f(1)+f(2)+f(3)+f(4)\right) = 15.890$

(b) Find the right Riemann sum using four subintervals of equal length.

$1\left(f(2)+f(3)+f(4)+f(5)\right) = 23.937$

(c) Find the trapezoidal Riemann sum using four subintervals of equal length.

$1\left[\frac{f(1)+f(2)}{2} + \frac{f(2)+f(3)}{2} + \frac{f(3)+f(4)}{2} + \frac{f(4)+f(5)}{2}\right] = 19.913$ or 19.914

(d) Find the midpoint Riemann sum using ~~two~~ subintervals of equal length.

$1\left[f(1.5)+f(2.5)+f(3.5)+f(4.5)\right] = 20.392$ or 20.393

put Y_1 = rate function then graph

13. (Calculator Active) A tank contains 10 gallons of water. Water is added at a rate of 4 gallons per minute, but neg. $\boxed{\text{leaks}}$ at a rate of $\sqrt{t}$ gallons per minute for time $t \geq 0$.

(a) How much is in the tank after 30 minutes? (FTC1 - or "net change" theorem")

$10 + \int_0^{30} 4 - \sqrt{t}\,dt = 20.455$ gallons ($\boxed{\text{MATH}}$ 9. fnInt)

(b) How long will it take to be down to 10 gallons again?

$10 + \int_0^{x} 4 - \sqrt{t}\,dt = 10$ | $10 + 4x - \frac{2}{3}x^{3/2} = 10$

$x = 36$ min.

(c) How long will it take for the tank to be empty?

$10 + \int_0^{x} 4 - \sqrt{t}\,dt = 0$ when $t = 40.57311552$ minutes

$10 + 4x - \frac{2}{3}x^{3/2} = 0$ (40 min 34.387 sec)

14. If $F(x) = \sqrt[3]{x+3}$ and $f(x) = F'(x)$, then $\int_5^{61} f(x)\,dx =$ $F(61) - F(5)$ (as F is an Antiderive of f)

(*Hint*: use FTC1)

$\sqrt[3]{64} - \sqrt[3]{8} = 4 - 2 = 2$

15. Let $F(x) = \int_0^{x^3} \sin t^2\,dt$. Use the second fundamental theorem of calculus to find $F'(x)$

$\sin(x^3)^2\,(3x^2)$ or $3x^2 \sin x^6$

16. Evaluate $\frac{d}{dx}\int_1^{4x^5} (t^2+t)^5\,dt$

$\left[(4x^5)^2 + (4x^5)\right] 20x^4$

use FTC2

17. A particle moves along a horizontal line so that its position at any time $t \geq 0$ is given by $s(t) = 2t^3 - 7t^2 + 4t + 5$, where s is measured in meters and t in seconds.

(a) (No Calc) Find the velocity function $v(t)$.

$$s'(t) = v(t) = 6t^2 - 14t + 4$$
$$= 2(3t^2 - 7t + 2)$$
$$= 2(3t-1)(t-2) \text{ meters per second}$$

(b) (No Calc) When is the particle at rest? Justify your answer

At rest when velocity is 0

$$2(3t-1)(t-2) = 0$$
$$t = \tfrac{1}{3} \text{ seconds}$$
$$\text{and } t = 2 \text{ seconds}$$

(c) (No Calc) When is the particle moving to the left? Justify your answer.

moves to left when s is decreasing, which is when velocity is negative

v: $+$ on $(0, \tfrac{1}{3})$, $-$ on $(\tfrac{1}{3}, 2)$, $+$ after 2

when t is between $\tfrac{1}{3}$ and 2 seconds

(d) (Calculator Active) If the particle starts at 1 at time $t = 0$, where will it be at time $t = 2.4$? *Show the "math" that you used, as well as the calculator result to 3 decimal places with units*

$$s(2.4) = s(0) + \int_0^{2.4} v(t)\,dt$$

$$\boxed{\begin{array}{l}\text{TI tip:} \\ \text{set } Y_1 = v(t) \\ \text{use } \boxed{\text{MATH}}\ 9.\text{fnInt}\end{array}}$$

$$1 + \int_0^{2.4} 6t^2 - 14t + 4\,dt = -2.072 \text{ meters}$$

The particle ended up at 2.072 meters to the left of 0.

(e) (Calculator Active) Find the total distance traveled by the particle between $t = 1$ and $t = 2$ *Show the "math" that you used, as well as the calculator result to 3 decimal places with units*

for an "odometer" distance, we ignore the sign of $v(t)$

$$\text{distance} = \int_1^2 |v(t)|\,dt = 3 \text{ meters}$$

The particle traveled 3 meters between 0 & 2 seconds

(f) (Calculator Active) What was the average <u>speed</u> of the particle between $t = 1$ to $t = 2$

$$\frac{1}{2-1}\int_1^2 |v(t)|\,dt = 3 \text{ meters per second}$$

Note this is a rate, not a distance

If you want the average rate from 0 to $t = 2.4$

$$\frac{1}{2.4-0}\int_0^{2.4} |v(t)|\,dt = 2.578 \text{ m/sec}$$

18. Without evaluating the integral, show how to use u substitution to express the indefinite integral $\int \frac{6t}{2t^2+5}\,dx$ in terms of u.

$u = 2t^2+5$

$du = 4t\,dx$

$\frac{3}{2}du = 6t\,dx$

so $\int \frac{6t}{2t^2+5}\,dx = \frac{3}{2}\int \frac{1}{u}\,du$

19. Without evaluating the integral, show how to use u substitution to express the indefinite integral $\int \frac{2x}{(4x+5)^3}\,dx$ in terms of u.

$u = 4x+5 \rightarrow x = \frac{u-5}{4}$

$du = 4\,dx$ $\quad 2x = \frac{u-5}{2}$

$\frac{1}{4}du = dx$

So $\int \frac{2x}{(4x+5)^3}\,dx = \frac{1}{4}\int \frac{u-5}{2}\cdot\frac{1}{u^3}\,du$

20. Without evaluating the integral, show how to use u substitution to express the definite integral $\int_1^e \frac{\ln x}{x}\,dx$ in terms of u.

$u = \ln x$ $\quad u(e) = \ln e = 1$

$du = \frac{1}{x}$ $\quad u(1) = \ln 1 = 0$

So $\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du$

21. $\int_1^5 (2x+1)(x^2+x)^3\,dx$

$u = x^2+x$

$du = 2x+1\,dx$

$u(1) = 1^2+1 = 2$

$u(5) = 5^2+5 = 30$

$\int_2^{30} u^3\,du = \frac{u^4}{4}\Big|_2^{30} = \frac{30^4}{4} - \frac{2^4}{4}$

$(= 202{,}496)$

22. $\int \frac{8x}{\sqrt{x^2+7}}\,dx$

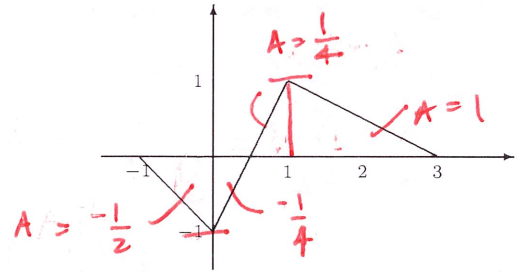
$u = x^2+7$

$du = 2x\,dx$

$4\,du = 8x\,dx$

$4\int u^{-1/2}\,du = 4\left[2u^{1/2}\right] = 8\sqrt{x^2+7} + C$

23. Let $g(x) = \int_{-1}^{x} f(t)\, dt$, where $f(t)$ is the function graphed below on the interval $[-1, 3]$.



Answer the following questions about the function g.

(a) (3 points)

i. $g(-1) =$ $\int_{-1}^{-1} f(t)\, dt = 0$

ii. $g(0) =$ $\int_{-1}^{0} f(t)\, dt = -\frac{1}{2}$

iii. $g(1) =$ $\int_{-1}^{1} f(t)\, dt = -\frac{1}{2} - \frac{1}{4} + \frac{1}{4} = -\frac{1}{2}$

(b) (3 points) On what interval is g increasing? Justify your answer.

Since $g' = f$

g': -1 ($-$) $\frac{1}{2}$ ($+$) 3

$g' > 0$ on $\left(\frac{1}{2}, 3\right)$

(c) (3 points) Does g have any relative extrema? If so, for what value(s) of x? Justify your answer.

$g'(x)$ changes from neg to pos @ $x = \frac{1}{2}$
so rel min at $g\left(\frac{1}{2}\right) = -\frac{3}{4}$

(d) (3 points) Does g have any points of inflection? If so, for what value(s) of x? Justify your answer.

POI is when there is a change of sign in g''
Since $g'' = f'$ we look for graph of f' Δ's
at $\left(0, -\frac{1}{2}\right)$ and at $\left(1, -\frac{1}{2}\right)$

24. Let $f(x) = \int_{-2}^{x} (t^3 - 27t)\, dt$. Determine all intervals on which f is concave down.

By FTC 2: $f'(x) = x^3 - 27x$

$f''(x) = 3x^2 - 27 = 3(x^2 - 9)$

f'': ($+$) -3 ($-$) 3 ($+$)

$f''(x) < 0$ on $(-3, 3)$

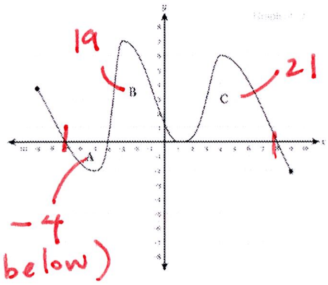
25. Evaluate $\int_{-1}^{1} 3(3x+1)^2\, dx$
Hint: Use u substitution

$\boxed{\begin{array}{ll} u = 3x+1 & u(-1) = -2 \\ du = 3\,dx & u(1) = 4 \\ \frac{1}{3}du = dx & \end{array}}$

$3 \cdot \frac{1}{3}\int_{-2}^{4} u^2\, du = \frac{u^3}{3}\Big|_{-2}^{4} = \frac{4^3}{3} - \frac{(-2)^3}{3}$

$= \frac{1}{3}(64+8) = \frac{72}{3} = 24$

26. The regions A, B, and C in the figure below are bounded by the graph of the function f and the x-axis. The area of region A is 4, the area of region B is 19, and the area of region C is 21. What is the average value of f on the interval $[-7,8]$?



$\frac{1}{8-(-7)}\int_{-7}^{8} f(x)\,dx = \frac{1}{15}[-4+19+21]$

$= \frac{36}{15} = \frac{12}{5} = 2.4$

27. A particle moves along the x-axis with velocity given by $v(t) = 4\pi\sin(2\pi t)$ for time $t \geq 0$. If the particle is at position $x = -3$ at time $t = \frac{2}{3}$, what is the position of the particle at time $t = \frac{1}{6}$?

$x = s(t) = \int v(t)\,dt = 2\int \sin u\, du = -2\cos(2\pi t) + C$

$-2\cos(2\pi \frac{2}{3}) + C = -3$

$C = -3-1 = -4$

$\boxed{\begin{array}{l} u = 2\pi t \\ du = 2\pi\, dt \\ 2\,du = 4\pi\, dt \end{array}}$

$s(t) = -2\cos(2\pi t) - 4$

$s(\frac{1}{6}) = -2\cos(2\pi \frac{1}{6}) - 4 = -1-4 = -5$

$\boxed{\text{Alt}}$ w/ TI-84 $\boxed{\text{MATH}}$ 9

$-3 + \int_{2/3}^{1/6} v(t)\,dt = -5$

28. Let $g'(x) = f(x)$ with $g(2) = a$, $g(3) = b$, $g(14) = c$, and $g(15) = d$.
Express the value of $\int_{-4}^{-2} xf(x^2-2)\,dx$ in terms of a, b, c, and/or d.

$\boxed{\begin{array}{l|l} \text{Let } u = x^2-2 & u(-4) = 14 \\ du = 2x\,dx & u(-2) = 2 \\ \frac{1}{2}du = x\,dx & \end{array}}$

$\frac{1}{2}\int_{14}^{2} f(u)\,du = -\frac{1}{2}\int_{2}^{14} f(u)\,du$

$= -\frac{1}{2}[g(14) - g(2)]$

$= -\frac{1}{2}[c - a]$ $(= \frac{1}{2}[a-c]$ BTW$)$